Moment of Inertia of Potentially Tilted Cuboids

A Draft Proof of Equation 10 at AAAI-15 paper:

"Towards Optimal Solar Tracking: a Dynamic Programming Approach"

Athanasios Aris Panagopoulos[a] Georgios Chalkiadakis[b]

February 14, 2015

Abstract

Calculating the moment of inertia of a tilted cuboid is essential for many practical applications, such as modeling a solar tracking system. In this paper we provide a *general* equation for calculating the moment of inertia of a potentially tilted, respecting its axes of rotation, cuboid.

1 General Cuboid Inertia Equation

The moment of inertia of a potentially tilted cuboid, I_c, can be calculated as:

$$I_c = \frac{m_c}{12}\left(l^2 \cos^2(\beta) + d^2 \sin^2(\beta) + w^2\right) \tag{1}$$

where m_c is the mass of the cuboid, β stands for its tilte angle, and l, w and d stand for its length, width and depth respectively. The dimensions and the tilte angle of the cuboid are all defined with respect to its axis of rotation, as seen in Figure 1.

2 Deriving the General Equation

In order to calculate the moment of inertia of a tilted cuboid, I_c, we calculated the moment of inertia of the imaginary cuboid that exactly contains the cuboid in question and we then subtracted the moment of inertia of the extra right angled prisms.

In particular the moment of inertia of the cuboid can be calculated through $I_c = I_{ALL} - (I_{R_1} + I_{R_2} + I_{R_3} + I_{R_4})$, where I_{ALL} stands for the moment of inertia of the imaginary cuboid, and I_{R_1}, I_{R_2}, I_{R_3} and I_{R_4} for the moment of inertia of the imaginary prisms, as illustrated in Figure 1. Note, however, that $I_{R_1} = I_{R_3}$ and $I_{R_2} = I_{R_4}$, and hence our calculation can be simplified as:

$$I_c = I_{ALL} - (2I_{R_3} + 2I_{R_4}) \tag{2}$$

[a]University of Southampton,UK, ap1e13@ecs.soton.ac.uk
[b]Technical University of Crete, Greece, gehalk@intelligence.tuc.gr

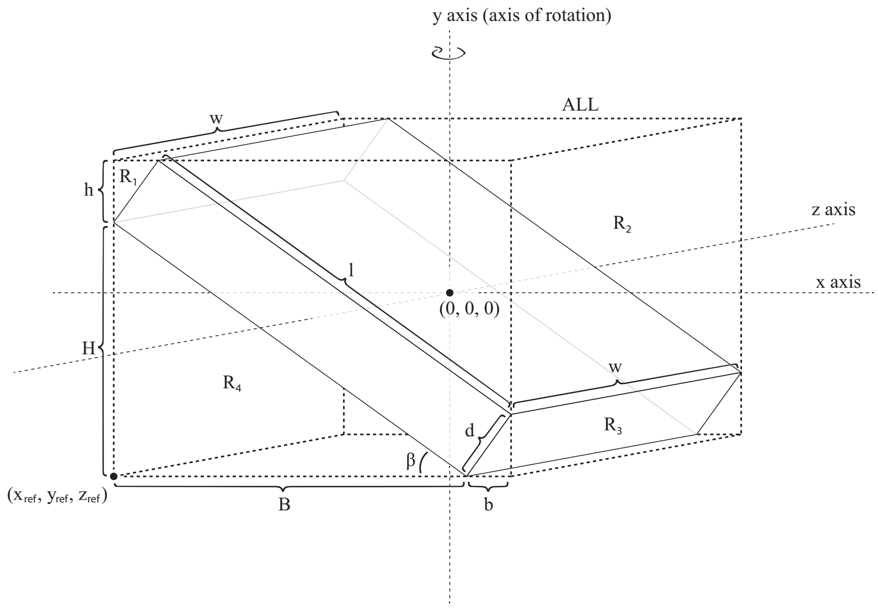

Figure 1: Tilted Cuboid

Please also note here as well, that this equation is suitable also for non-tilted cuboids, as in that case $I_{R_1} = I_{R_2} = I_{R_3} = I_{R_4} = 0$ and hence $I_c = I_{All}$ (and hence so does Eq 1). In the following paragraphs we calculate the I_{ALL}, I_{R_3} and I_{R_4} above.

2.1 I_{ALL}

I_{ALL} can be calculated from the non-tilted cuboid Equation 3 [1].

$$I_{ALL} = \frac{m_{ALL}}{12}\left((B+b)^2 + w^2\right) \tag{3}$$

where m_{ALL} is the mass of the imaginary cuboid and the quantity $(B+b)$ stands for the length of the imaginary cuboid, as seen in Figure 1. The mass of the imaginary cuboid can be calculated based on its volume; V_{ALL}, and its density; ρ, via $m_{ALL} = \rho V_{ALL}$ The density of the imaginary cuboid will appropriately be the same as the density of the cuboid in question and can be calculated as $\rho = \frac{m_c}{lwd}$. The volume can be calculated as $V_{ALL} = (B+b)(H+h)w$, where the quantity $(H+h)$ stands for the depth of the imaginary cuboid, as seen in Figure 1. H, h, B and b can be appropriately computed based on the dimensions and the tilt angle of the cuboid in question through Equations 4, 5, 6 and 7, respectively.

$$H = sin(\beta)l \tag{4}$$

$$h = cos(\beta)d \tag{5}$$

$$B = cos(\beta)l \tag{6}$$

$$b = sin(\beta)d \tag{7}$$

2.2 I_{R_3}

The moment of inertia of the right angled prism R_3 with respect to an axis of rotation that passes through its centroid and is parallel to the axis of rotation of the cuboid in question can be calculated through Equation 8 [1] below.

$$I_{R_3}^{centroid} = \frac{m_{R_3}}{36}(2b^2 + 3w^2) \tag{8}$$

where m_{R_3} is the mass of the prism which can be calculated as $m_{R_3} = \frac{bhw}{2}\rho$.

Now the centroid of the right angled prism is defined as:

$$(-(x_{ref} + \tfrac{B}{3}),\ -(y_{ref} + \tfrac{H}{3}),\ -(z_{ref} + \tfrac{w}{2}))$$

As such, from Equation 8 and by applying the parallel axes theorem [2], we will have:

$$I_{R_3} = \frac{m_{R_3}}{36}(2b^2 + 3w^2) + m_{R_3} D_{R_3}^2 \tag{9}$$

where D_{R_3} is the distance of the centroid from the axis of rotation:

$$D_{R_3} = \frac{B + b}{2} - \frac{b}{3} \tag{10}$$

2.3 I_{R_4}

With the same method in computing I_{R_3} above, we will have:

$$I_{R_4} = \frac{m_{R_4}}{36}(2B^2 + 3w^2) + m_{R_4} D_{R_4}^2 \tag{11}$$

where $m_{R_4} = \frac{BHw}{2}\rho$ and for D_{R_4}:

$$D_{R_4} = \frac{B + b}{2} - \frac{B}{3} \tag{12}$$

2.4 I_c

By combining Equations 2, 9, 11 and 3, and by applying simple arithmetic operations we derive the general Equation 1 for calculating the moment of inertia for potentially tilted cuboids.

References

[1] J.A. Myers. *Handbook of Equations for Mass and Area Properties of Various Geometrical Shapes.* NAVWEPS report 7827. U.S. Naval Ordnance Test Station, 1962.

[2] T.R. Kane and D.A. Levinson. *Dynamics: Theory and Applications.* McGraw-Hill Series in Mechanical Engineering. McGraw-Hill Ryerson, Limited, 1985.